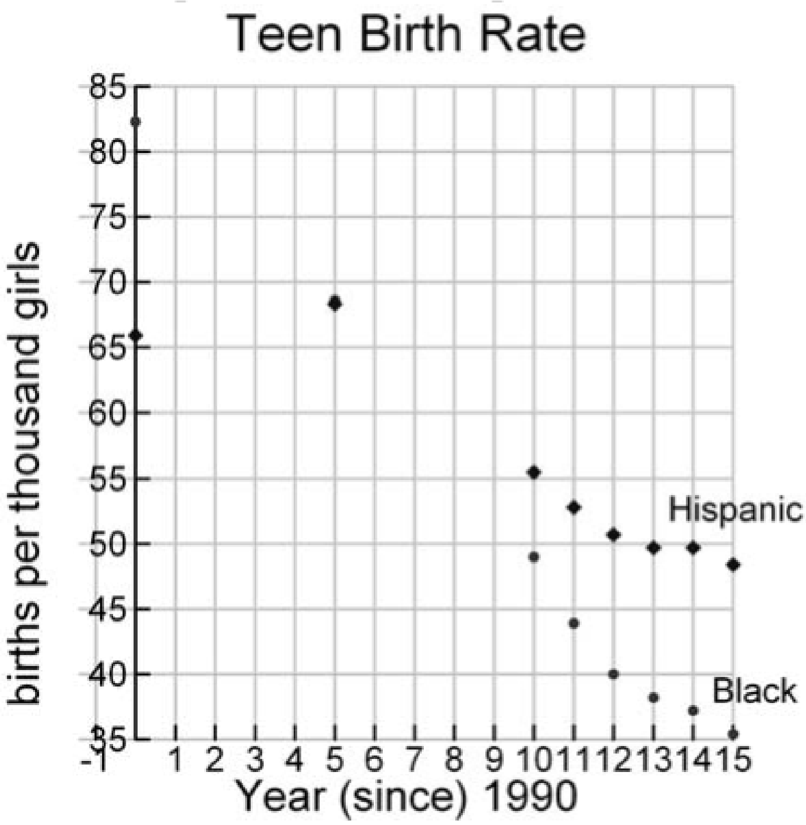
Teen Pregnancy

This scatter plots show the pregnancy rate of 15-17 year old young women racially classified as Black and Hispanic in an English speaking country.

The birth rate is reported as births per 1000 women.



Tasks :

1. Describe the situation which is developed in the document.

2. Explain from the scatter plots if it appears that efforts to educate youth about abstinence and safe sex are having an impact.

3. A scientist suggests two models, the cubic function model for each of the data sets is given below.
Hispanic birth rate : $h(x) = 0.0280x^3 – 0.718x^2 + 3.34x + 65.9$;
Black birth rate : $b(x) = 0.0215x^3 – 0.475x^2 - 0.846x + 82.3$.
Enter the two functions in your graph calculator and comment on the reliability of the models.

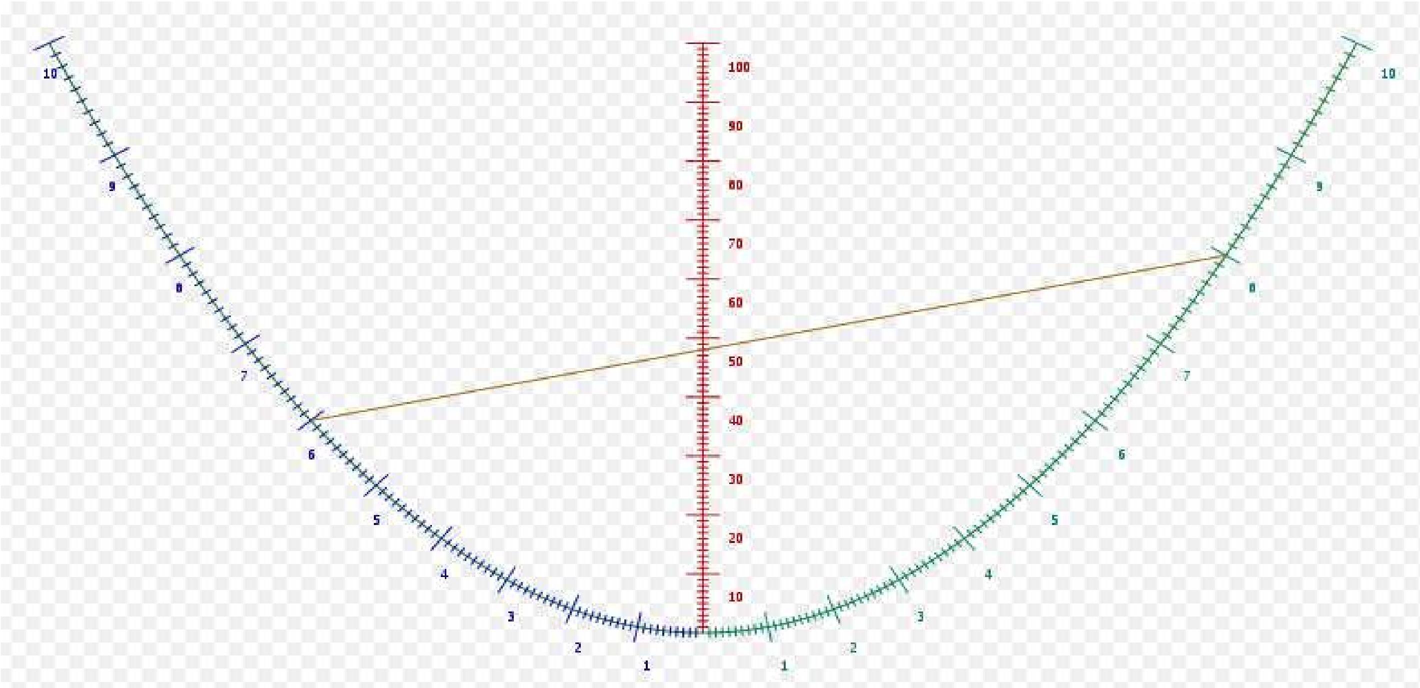
Source : www.makeitreallearning.com

The multiplication parabola

A nomogram, also called an abacus, is a graphical calculating device designed to allow the approximate graphical computation of a function.

The following nomogram is based on the graph of a parabola.

Choose point 6 on one half of the parabola and point 8 on the other half of the parabola. Then draw the line and note the intersection point with the vertical axis : that's 48... which is 6 × 8 !



Adapted from wikipedia

Tasks

1. Choose two integers and verify that their product can be found on the above graph.
2. Consider the parabola $y = x^2$. State the properties of this elementary function and the properties of its graph.
3. Let a and b be two distinct real numbers.
   a) Compute the slope of the line that crosses the parabola $y = x^2$ where $x = a$ and where $x = b$ on the parabola.
   b) Deduce the equation of the line.
   c) Compute its y-intercept.
   d) Conclude.
4. Explain the construction of the nomogram from the graph of the parabola $y = x^2$.
5. Let's consider two integers a and b greater than or equal to 2 on both sides of the parabola, join them and look closely at the intersection point with the vertical axis.
   a) Explain why 3 and 5 will not be reached.
   b) Give some examples of other integers that will not be reached.
   c) Give the major arithmetical property that all these integers share.

The multiplication parabola
Comments and answers

1. Choose any pair of integers on both halves of the parabola : the line crosses the y-axis at the value of their product !

2. The parabola $y = x^2$ is the graph of the squaring function, which is an even function, with domain R (set of all real numbers) and range the interval $[0;+\infty)$.
   It is symmetrical about the y-axis, in a U-shape (or upwards), with the vertex at point (0,0).

3. Let's consider a and b be two distinct real numbers.
   a) Let's call A the point $(a; a^2)$ and B the point $(b; b^2)$.
   The gradient of line (AB) is : $\boxed{\dfrac{y_B - y_A}{x_B - x_A} = \dfrac{b^2 - a^2}{b-a} = \dfrac{(b-a)(b+a)}{b-a} = b+a}$.
   b) Then the equation of line (AB) is : $y - a^2 = (a + b)(x - a)$ or $y - b^2 = (a + b)(x - b)$, or $\boxed{y = (a + b)x - ab}$.
   c) The y-intercept is the value of y when $x = 0$ which is here $\boxed{-ab}$.
   d) We have proved that the y-intercept of the line joining two points on the graph of the parabola $y = x^2$ is <u>the opposite of the product of their abscissas</u>.

4. To build the nomogram from the graph of the parabola $y = x^2$, we have to mark absolute values of abscissas on each half of the parabola, both sides of the y-axis.

5. a) If a and b are both greater than or equal to 2, then, their product is an integer greater than or equal to 4. Then 3 can't be reached on the vertical axis. The integer 5 can't be reached as well because 5 is not the product of two integers.
   b) That's also the case of 7, 11, 13, 17...etc which are all integers that are not the product of two integers.
   c) Such integers are called <u>prime numbers</u> that are divisible only by 1 and themselves.

How can numbers have a geometric shape?

Although numbers do not have a geometric shape, some can be represented by dots that can be put into a geometric shape. Here are some examples.

Triangular numbers:

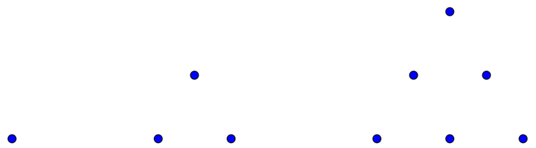


Diagram 1 Diagram 2 Diagram 3

Oblong numbers:

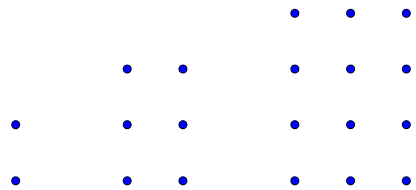


Diagram 1 Diagram 2 Diagram 3

Tasks:

1. Make a sketch of diagram 4 in each sequence.
   Explain how you made them.

2. Complete the empty squares in the table for diagrams 4 and 5.

Diagram n	1	2	3	4	5	n
Triangular number: number of dots	1	3	6			Tn
Oblong number: number of dots	2	6	12			On

3. Using the table above, say what conclusion you can draw.

4. Explain: "Tn is the sum of the n first whole numbers".

5. Can you conjecture an equivalent property of oblong numbers?

6. Prove that :

   a) $Tn = \dfrac{n\,(n+1)}{2}$

   b) $On = n\,(n+1)$

7. The expressions of Tn and On confirm what is found in questions 3 and 5. Discuss.

Baccalauréat général – Session 2012 Académie de Caen
Section européenne, DNL Mathématiques – Anglais
Sujet 4

The Cantor set.

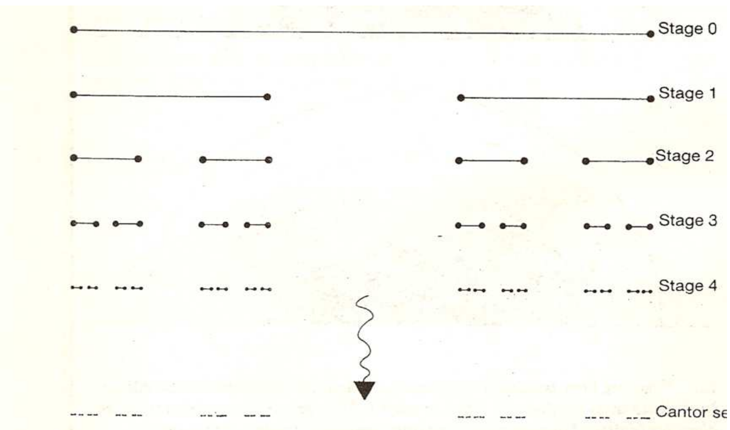
(This is a conversation between Henry and Anne-Lida.)

- *Henry* : Georg Cantor was a German mathematician who invented a very curious set in about 1883 […]To get a Cantor set you start with a line segment of length 1, and remove its middle third. Now remove the middle third of each remaining piece. Repeat, forever. What is left is the Cantor set. (Figure 1.6)
- *Anne-Lida* : I don't see how there can be *anything* left, Henry.
- *Henry* :Oh, but there is. All the end-points of all the smaller segments are left, for a start. And many others. But you are right in one way, my dear. What is the length of the Cantor set ?
- *Anne-Lida* :Its ends are distance 1 apart, Henry.
- *Henry* :No, I meant the length not counting the gaps.
- *Anne-Lida* :I have no idea, Henry. But it looks very small to me. The set is mostly holes.
- […]
- *Henry* :The length reduces to 2/3 the size at each stage, so the total length after the *n*th stage is $\left(\frac{2}{3}\right)^n$. As *n* tends to infinity, this tends to 0. The length of the Cantor set is zero.

Ian Stewart, « Game, Set & Math, enigmas and conundrums », Dover publications, 2007.

Stage	Segment length	Number of segments	Total set length	Number of end-points
0	1	1	1	2
1	$\frac{1}{3}$	2	$2\times\frac{1}{3}=\frac{2}{3}$	2^2
2				
3				
4				

Table 1



1.6 Construction of the Cantor set by removing middle thirds. Its length is zero, but contains infinitely many points.

TASKS :

1) Fill in table 1 above with integers or fractions raised to the appropriate power.
2) Conjecture the line of table 1 at stage n.
3) Explain why the Cantor set at stage n contains more than $2^{(n+1)}$ points.
4) Explain why the Cantor set at stage n has length $\left(\frac{2}{3}\right)^n$.
5) Let n go to infinity and give the length of the Cantor set. Justify your answer.
6) How many points are contained in this Cantor set ?
7) Using the preceeding results, do you think it is possible for a set to be simultaneously of length zero and non empty ? Explain your reasoning.

The Cantor set.
ANSWERS

1)

Stage	Segment length	Number of segments	Number of end-points	Total set length
0	1	1	2	1
1	1/3	2	2^2	2*1/3=2/3
2	1/3*1/3=(1/ 3)^2	2*2=2^2	2^3	2^2*(1/3)^2=(2/3)^2
3	1/3*(1/3)^2=(1:3)^3	2^3	2^4	2^3*(1/3)^3=(2/3)^3
4	1/3*(1/3)^3=(1/3)^4	2^4	2^5	(2/3)^4

…..
2)
n (1/3)^n 2^n 2^(n+1) (2/3)^n

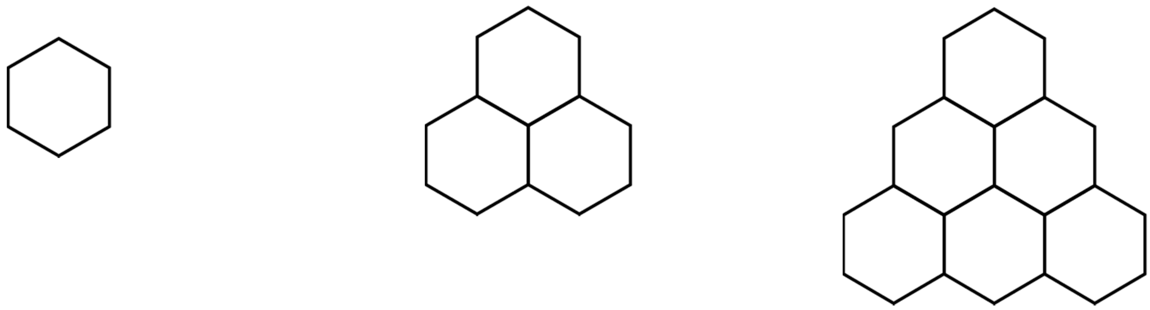
3) the number of end-points is doubled at each stage starting with 2 points at stage 0. Therefore, this number of points is a geometric sequence with constant 2 and 2 as first term. At stage n, the number of end-points contained in the set is 2^(n+1) and the set contains the points of the small line segments determined by these end-points too.

4) The length of the Cantor set is
(the number of line segments) x (the segment length)
= 2^n x (1/3)^n
= (2/3)^n

5) As n goes to infinity, the Cantor set has length zero, using the limit of a geometric sequence whose constant q satisfies 0<q<1

6) As n goes to infinity, 2^(n+1) goes to infinity, since 2>1. Therefore, the Cantor set contains infinetely many points.

7) Thus, the Cantor set shows that it is possible for a set to be of length 0 and non empty, which is contradicting our common sense. This set contains infinitely many points since 2^(n+1) goes to infinity when n goes to infinity.



Step 1 Step 2 Step 3

Jeremy a bee-keeper is doing an investigation which involves making patterns of hexagons with sticks. The first of these three patterns are shown above.

Tasks

1. Describe the easiest way to find the extra number of hexagons and the extra number of sticks necessary in step 4.

2. Copy and complete this table

number of rows, n	1	2	3	4
number of hexagons, h	1	3		
number of sticks, S	6			

3. Jeremy doesn't know which formula to use to calculate S in terms of n.

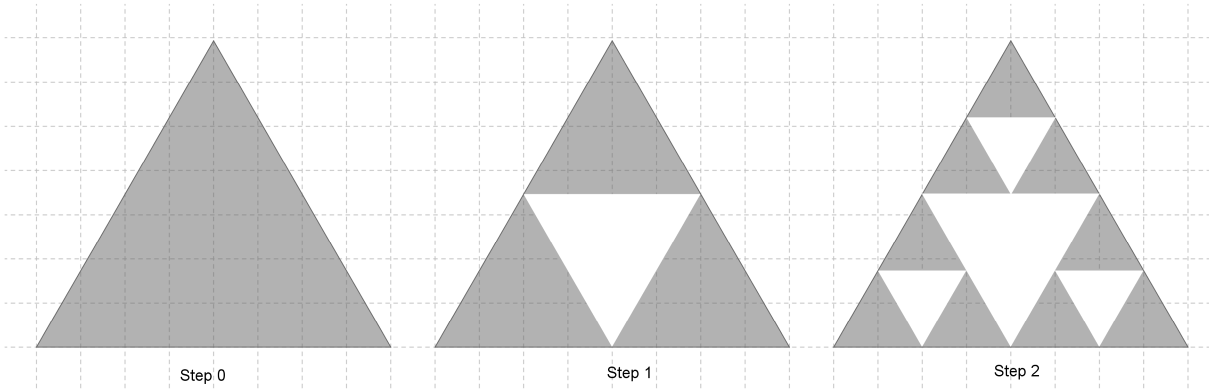
$S = 2n^2 + 4n$ $\qquad$ $S = \frac{3}{2}n^2 + \frac{9}{2}n$ $\qquad$ $S = \frac{5}{2}n^2 + \frac{3}{2}n + 2$

   (a) Help him to choose the right formula.
   (b) Explain the reasons why you have arrived at this result.

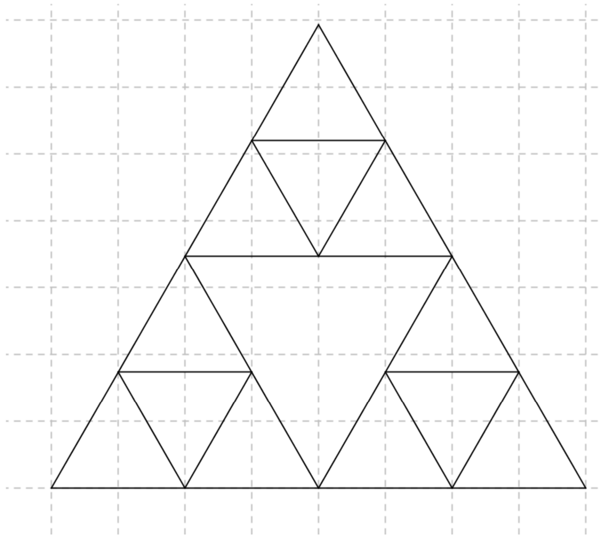
4. Find the number of rows for the number of sticks to be 105.

Fractals are geometric shapes that are self-similar at different scale. They are formed by applying the same procedure over and over again. Sierpinski's triangle is one of the most famous examples of a fractal, named after the Polish mathematician **Waclaw Sierpinski** (1882-1969) who described it in 1915.

Sierpinski's triangle is created by infinitely repeating the construction process shown below. In step 0, an equilateral triangle is drawn.



The goal is to look at the pattern and to find relations between the step number and the different geometric elements.



Tasks :

1°) Describe the procedure to construct the next step. (You could use the adjoining figure.)

2°) **Number of remaining triangles (grey ones).**
In step 0, there is one triangle remaining.

How many triangles remain in step 1, step 2 ?

Explain the pattern

3°) **Area of Sierpinski's triangle. (grey area).**
Let the area of the original triangle be 1 (triangular unit).

In step 1, justify that the area of Sierpinski's triangle is ¾.

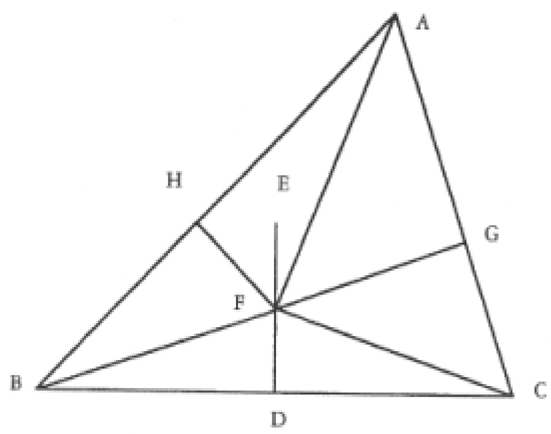
What is the area of the Sierpinski's triangle in step 2 ?

Deduce the formula for the area of the Sierpinski's triangle for any step.

4°**)** What can you say about the number of triangles and the area of Sierpinski's triangle by repeating the process infinitely ?

All triangles are isosceles...

The English author Lewis Carroll (1832 –1898) found fame after the publication of his book *Alice's Adventures in Wonderland*. But less well known, is the “other” Lewis Carroll, the mathematician at Christ Church College, Oxford. Within the academic discipline of mathematics, he worked primarily in the fields of recreational mathematics, producing a dozen books. In one of these, he developped the following proof that every triangle has two equal sides :



Let ABC be any triangle. Bisect BC at D and from D draw DE at right angles to BC. Draw line AF that bisects $\angle$BAC. Join FB, FC, and from F, draw FG, FH, at right angles to AC, AB.

(1) $\angle$FGA = $\angle$FHA;

(2) $\angle$FAG = $\angle$FAH;

(3) Triangles AFG and AFH have the side AF in common;

(4) Therefore, AG = AH.

(5) Similarly, we have FG = FH.

(6) FB = FC.

(7) Therefore, combining steps (5) and (6), we have GC = HB.

(8) Combining steps (4) and (7), AC = AB.

Therefore, triangle ABC is isosceles.

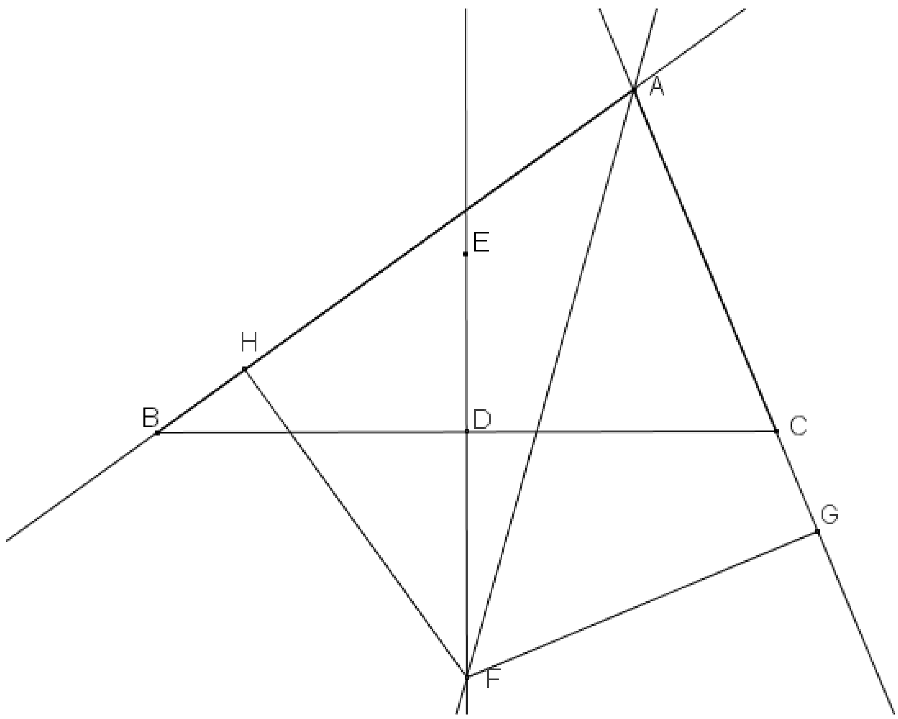
Adapted from Rediscovered Lewis Carrol puzzles by Edward Wakeling

Tasks

1. Propose a short summary of the subject and give your first impression.
2. Line DE is known as a remarkable line in triangle ABC : give some more information about such a line.
3. Line AF is known as a remarkable line in triangle ABC : give some more information about such a line.
4. Resume step by step Caroll’s demonstration and justify each statement.
5. By drawing your own picture accurately, find out the fallacy in Caroll’s proof.
6. Talk about any other famous English mathematicians you’ve heard of.

All triangles are isosceles...

A donner éventuellement au candidat lorsqu'il aborde la question 5

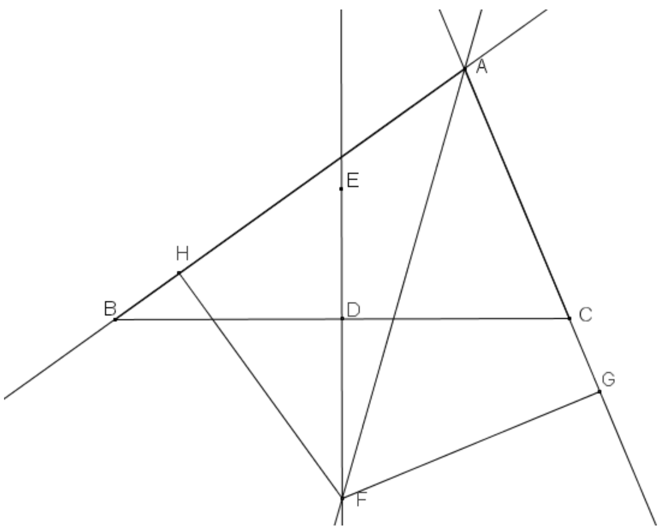


All triangles are isosceles...

Comments and answers

1. The document is proposing a geometrical proof of a false situation given by Lewis Carrol.
2. Line DE is the perpendicular bisector of line segment BC. It's the locus of points in the plane that are the same distance from points B and C.The three perpendicular bisectors of the sides of a triangle are concurrhgent and intersect at a point called the circumcentre, or the centre of the the circumscribing circle of the triangle.
3. Line AF is the angle bisector of angle BAC. It's the locus of points in the plane that are the same distance from lines AB and AC. The three angle bisectors of the internal angles of a triangle are concurrent and intersect at a point called the incentre, or the centre of the incircle of the triangle, the circle that is tangent to the triangle.
4. Step by step justifications:
   (1) $\angle$FGA = $\angle$FHA because they're both right angles by construction of points G and H;
   (2) $\angle$FAG = $\angle$FAH because line AF bisects angle BAC;
   (3) Triangles AFG and AFH have the side AF in common which is obvious !;
   (4) In right triangle AFG with right angle at point G, AG = AF $\times \cos F\hat{A}G$, and in right triangle AFH with right angle at point H, AH = AF $\times \cos F\hat{A}H$. Thus, using (2), AG = AH;
   (5) Similarly, we have FG = FH since FG = AF $\times \sin F\hat{A}G$ and FH = AF $\times \sin F\hat{A}H$;
   (6) FB = FC since F belongs to the perpendicular bisector of line segment BC;
   (7) Using the Pythagoras theorem in right triangles FGC, FHB, with right angle at point G, H, we have : $GC^2 = FC^2 - FG^2$ and $HB^2 = FB^2 - FH^2$, leading to $GC^2 = HB^2$ from (5) and (6), and therefore, GC = HB;
   (8) Combining steps (4) and (7), AC = AG + GC = AH + HB = AB;

   Therefore, triangle ABC is isosceles since it has two equal sides.
5. An accurate drawing shows that point F is outside the triangle… Thus, one of the points G, H is also outside the triangle, so that the statement AC = AG + GC and AB = AH + HB is false : on the following picture, we still have AB = AH + HB but AC = AG – GC, and not AG + GC !



6. Newton and his works concerning the derivative of a function in the 17^{th} century.